REAL ANALYSIS
TOPIC VII: CARDINALITY

PAUL L. BAILEY

Abstract. We have seen that there are infinitely many rational numbers and infinitely many irrational numbers on the real line. In this chapter we discuss how there are actually more irrational numbers than rational numbers, and attempt to understand exactly how many real numbers there are.

1. Infinite Sets

Axiom 1 (Axiom of Choice)**.** *The cartesian product of a collection of nonempty sets is nonempty.*

Theorem 1 (Zorn's Lemma)**.** *Let A be a partially ordered set. If every chain in A has an upper bound, then A contains a maximal element.*

Remark. Zorn's Lemma can be proven using the Axiom of Choice, and in fact is logically equivalent to it. It may be used to show many useful set-theoretic propositions, including the following. □

Proposition 1. *Let A and B be sets. Then there exists either a surjective function $A \to B$ or a surjective function $B \to A$.*

Proposition 2. *Let A and B be sets. Then there exists an injective function $A \to B$ if and only if there exists a surjective function $B \to A$.*

Theorem 2 (Schroeder-Bernstein Theorem)**.** *Let A and B be sets. If there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$.*

Remark. We include a proof of this at the end of the chapter. □

Definition 1. Let $\mathbb{N} = \{1, 2, \dots\}$. Let $n \in \mathbb{N}$ and set $\mathbb{N}_n = \{1, \dots, n\}$.

Observation 1. Let $f : \mathbb{N}_n \to \mathbb{N}_n$. Then f is injective if and only if f is surjective.

Observation 2. Let $f : \mathbb{N} \to \mathbb{N}_n$ be a function. Then f is not injective.

Definition 2. Let A be a set. We say that A is *finite* there exists a bijective function $\mathbb{N}_n \to A$ for some $n \in \mathbb{N}$. We say that A is *infinite* if there exists an injective function $\mathbb{N} \to A$.

Proposition 3. *Let A be a set. Then A is infinite if and only if it is not finite.*

Date: October 29, 2018.

2. Countable Sets

Definition 3. Let A be a set. We say that A is *countable* if there exists a surjective function $\mathbb{N} \to A$.

Proposition 4. *Every subset of a countable set is countable.*

Proof. Let A be a countable set and let $B \subset A$. Since A is countable, there exists an injective function $f : A \to \mathbb{N}$. Then $f \restriction_B : B \to \mathbb{N}$ is also injective, so B is countable. $\square$

Proposition 5. *Let A and B be countable sets. Then $A \cup B$ is countable.*

Proof. Since A and B are countable, there exist surjective functions $g : \mathbb{N} \to A$ and $h : \mathbb{N} \to B$. Define a function

$$f : \mathbb{N} \to A \cup B \quad \text{by} \quad f(n) = \begin{cases} g(\frac{n}{2}) & \text{if } n \text{ is even;} \\ h(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Then f is surjective, so $A \cup B$ is countable. $\square$

Proposition 6. *Let A and B be countable sets. Then $A \times B$ is countable.*

Proof. Since A and B are countable, there exist injective functions $g : A \to \mathbb{N}$ and $h : B \to \mathbb{N}$. Define a function

$$f : A \times B \to \mathbb{N} \quad \text{by} \quad f(a, b) = 2^{g(a)} \cdot 3^{h(b)}.$$

To see that f is injective, suppose that $f(a_1, b_1) = f(a_2, b_2)$. Then $2^{g(a_1)}3^{h(b_1)} = 2^{g(a_2)}3^{h(b_2)}$. Thus $2^{g(a_1)-g(a_2)} = 3^{h(b_2)-h(b_1)}$, where without loss of generality $g(a_1) \geq g(a_2)$. If $g(a_1) > g(a_2)$, then 2 divides the left side and not the right; this is impossible, so $g(a_1) = g(a_2)$, and since g is injective, we must have $a_1 = a_2$. Similarly, $b_1 = b_2$. $\square$

Proposition 7. *The set $\mathbb{N}$ of natural numbers is countable.*

Proof. The identity function $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is surjective. $\square$

Proposition 8. *The set $\mathbb{Z}$ of integers is countable.*

Proof. Define a function

$$f : \mathbb{Z} \to \mathbb{N} \quad \text{by} \quad f(n) = \begin{cases} 1 & \text{if } n = 0; \\ 2n & \text{if } n > 0; \\ 2n + 1 & \text{if } n < 0. \end{cases}$$

Then f is injective, so $\mathbb{Z}$ is countable. $\square$

Proposition 9. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. By Proposition 6, it suffices to find an injective function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$. Every rational number has a unique expression $\frac{p}{q}$ as a ratio of integers, where $\gcd(p, q) = 1$ and $q > 0$. This induces a function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ given by $\frac{p}{q} \mapsto (p, q)$. This function is bijective; therefore $\mathbb{Q}$ is countable. $\square$

3. Base β Expansions

Let A be a set. A *sequence* in A is a function $a : \mathbb{N} \to A$. We write a_i to mean $a(i)$, and we write $(a_i)_{i=1}^{\infty}$, or simply (a_i), to denote the function a. Let $\mathcal{S}(A)$ denote the set of all sequences in A.

Let β be an integer with $\beta \geq 2$, and let $\mathbb{Z}_\beta = \{0, 1, \dots, \beta - 1\}$. Let $O = (0, 1)$ be the open unit interval in the real line. We are interested in relating the set of sequences in $\mathbb{Z}_\beta$, which we denote by $\mathcal{S}(\mathbb{Z}_\beta)$, to the set O.

Define a function

$$\mu : \mathbb{Z} \to \mathbb{Z}_\beta \quad \text{by} \quad \mu(n) = r,$$

where $n = \beta q + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < \beta$.

Define a function

$$\zeta : \mathbb{R} \to \mathbb{Z} \quad \text{by} \quad \zeta(x) = \max\{n \in \mathbb{N} \mid n \leq x\}.$$

For each $k \in \mathbb{N}$, define a function

$$\delta_{\beta,k} : \mathbb{R} \to \mathbb{Z}_\beta \quad \text{by} \quad \delta_{\beta,k}(x) = \mu(\zeta(\beta^k x)).$$

This induces a function

$$\delta_\beta : O \to \mathcal{S}(\mathbb{Z}_\beta) \quad \text{by} \quad \delta_\beta(x) = (\delta_{\beta,k}(x))_{k=1}^{\infty}.$$

Then δ_β is an injective function, and we call $\delta_\beta(x)$ the *base β expansion* of x.

Construct a partial inverse to δ_β as follows.

Let $\{a_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{Z}_\beta$ and set $B = \{\sum_{i=1}^{k} \frac{a_i}{\beta^i} \mid k \in \mathbb{N}\}$. Then $B \subset O$, and in particular, B is a bounded set of real numbers. Set $b = \sup(B)$. For most sequences, $\delta_\beta(b) = (a_i)_{i=1}^{\infty}$.

Call a sequence $(a_i)_{i=1}^{\infty}$ in $\mathbb{Z}_\beta$ a *duplicator* if there exists $N \in \mathbb{N}$ such that $a_i = \beta - 1$ for all $i > N$. These are the only sequences which are not in the image of the function δ_β. If $S = \mathcal{S}(\mathbb{Z}_\beta) \smallsetminus \{ \text{ duplicators } \}$, then $\delta_\beta : O \to S$ is bijective.

4. Uncountability

Proposition 10. *The set $\mathbb{R}$ of real numbers is an uncountable set.*

Proof. Since $O = (0, 1) \subset \mathbb{R}$, it suffices to show that O is uncountable.

Let $\beta = 10$ so that we consider base 10 expansions of the elements in O, and let μ, ζ, and δ_β be as in the previous section.

Let $f : \mathbb{N} \to O$ be any function; we will show that f is not surjective. Define a sequence (a_i) in $\mathbb{Z}_{10}$ by

$$a_i = \begin{cases} 3 & \text{if} \quad \delta_{\beta,i}(f(i)) \neq 3; \\ 6 & \text{if} \quad \delta_{\beta,i}(f(i)) = 3. \end{cases}$$

Define a sequence (s_n) by

$$s_n = \sum_{i=1}^{n} \frac{a_i}{10^i}.$$

Then (s_n) is a bounded increasing sequence; let $b = \lim_{n\to\infty} s_n$. Clearly $\delta_\beta(b) = (a_i)$, and b is not in the image of f. $\square$

5. Measure

In this section, we briefly describe the concepts of measure and probability, and say why the probability of selecting a rational number from the closed unit interval is zero.

Definition 4. Let I be an interval. The *length* of I, denoted $L(I)$, is the distance between its endpoints. Then $L(I)$ is a nonnegative extended real number.

Let $A \subset \mathbb{R}$. An *open interval cover* of A is an open cover of A consisting of open intervals. Let $\mathcal{O}$ be an open interval cover of A. Define the *length* of $\mathcal{O}$, denoted to be the sum of the lengths of the intervals in $\mathcal{O}$:

$$L(\mathcal{O}) = \sum_{I \in \mathcal{O}} L(I).$$

The *outer measure* of A is

$$m(A) = \inf\{L(\mathcal{O}) \mid \mathcal{O} \text{ is an open interval cover of } A\}.$$

We say that A is *measurable* if

$$m(A) = m(A \smallsetminus B) + m(B) \quad \text{for all} \quad B \subset \mathbb{R}.$$

Let $B \subset A$. The *probability* of selected an element of B from A is

$$P(B|A) = \frac{m(B)}{m(A)}.$$

Ponder the following facts:

(a) if I is an interval, then I is measurable, and $m(I) = L(I)$;
(b) if A and B are measurable, then $A \cup B$, $A \cap B$, and $A \smallsetminus B$ are measurable;
(c) if $m(A) = 0$, then A is measurable.

Let $U = [0, 1]$; then $m(U) = 1$. Set $Q = U \cap \mathbb{Q}$. We would like to show that $P(Q|U) = 0$. To do this, we only need to show that $m(Q) = 0$.

Proposition 11. *Let $U = [0, 1]$ and $Q = U \cap \mathbb{Q}$. Then $m(Q) = 0$.*

Proof. Since $\mathbb{Q}$ is countable, so is Q. Let $q : \mathbb{N} \to Q$ be surjective, and let $q_n = q(n)$; in this way, think of (q_n) as a sequence in U whose image in Q.

Let r be a positive real number. Set $rI_n = (q_n - \frac{r}{2^{n+1}}, q_n + \frac{r}{2^{n+1}})$. Let $r\mathcal{O} = \{rI_n \mid n \in \mathbb{N}\}$. Then

$$L(r\mathcal{O}) = \sum_{n=1}^{\infty} \frac{r}{2^{n+1}} = r.$$

Let $\epsilon > 0$ and let $r = \frac{\epsilon}{2}$. Then $0 < L(r\mathcal{O}) < \epsilon$. Since this is true for every $\epsilon > 0$, we have $m(Q) = 0$. □

6. Cardinality

Definition 5. Let A and B be sets. We say that A and B have the *cardinality* if there exists a bijective function $A \to B$.

If A and B have the same cardinality, we write $|A| = |B|$. If there exists an injective function $A \to B$, we write $|A| \leq |B|$. If there does not exist a surjective function $A \to B$, we write $|A| < |B|$.

Definition 6. Let A be a set. The *power set* of A, denoted $\mathcal{P}(A)$, is the collection of all subsets of A.

Proposition 12. *Let X be a set. Then $|X| < |\mathcal{P}(X)|$.*

Proof. Let $f : X \to \mathcal{P}(X)$; we wish to show that f is not surjective. Set

$$Y = \{x \in X \mid x \notin f(x)\}.$$

Suppose, by way of contradiction, that $f(x) = Y$ for some $x \in X$. Is $x \in Y$? If it is, then $x \in f(x)$, so by definition of Y, $x \notin Y$. On the other hand, if it is not, then $x \notin f(x)$, so $x \in Y$. Either case is an immediate contradiction. Thus there is no such x satisfying $f(x) = Y$, and Y is not in the image of f. Therefore f is not surjective. $\square$

Remark 1. This shows that for every set, including infinite sets, there is always a set with a larger cardinality. The cardinality of an infinite set may be regarded as a level of infinity, and in this way, there are infinitely many levels of infinity.

Definition 7. Let A and B be sets. Let $\mathcal{F}(A, B)$ denote the set of all functions from A to B.

Proposition 13. *Let X be any set. Then $|\mathcal{P}(X)| = |\mathcal{F}(X, \mathbb{Z}_2)|$.*

Proof. Define a function

$$\Phi : \mathcal{F}(X, \mathbb{Z}_2) \to \mathcal{P}(X) \quad \text{by} \quad \Phi(f) = f^{-1}(1).$$

It suffices to show that Φ is bijective.

To see that Φ is injective, suppose that $\Phi(f_1) = \Phi(f_2)$, where $f_1 : X \to T$ and $f_2 : X \to T$. Then $f_1(x) = 1$ if and only if $f_2(x) = 1$. For $x \in X$, $f_i(x)$ is either 1 or 0, so if it is not 1, it is zero. Therefore $f_1(x) = 0$ if and only if $f_2(x) = 0$. So $f_1(x) = f_2(x)$ for every $x \in X$, that is, $f_1 = f_2$.

To see that Φ is surjective, let $A \in \mathcal{P}(X)$. Define a function

$$f : X \to \mathbb{Z}_2 \quad \text{by} \quad f(x) = \begin{cases} 0 & \text{if } x \notin A; \\ 1 & \text{if } x \in A. \end{cases}$$

Then $A = f^{-1}(1)$, so $\Phi(f) = A$. $\square$

7. Intervals

Proposition 14. *Any two intervals have the same cardinality.*

Proof. We show part of this and leave the remaining details to the reader.

First note that the function $x \mapsto \frac{x-a}{b-a}$ maps (a,b) bijectively onto $(0,1)$. So all intervals of type **(a)** have the same cardinality.

To see that a closed interval has the same cardinality as an open interval, consider the function $f : [0,1] \to (0,1)$ given by

$$f(x) = \begin{cases} 1/2 & \text{if } x = 0; \\ 1/3 & \text{if } x = 1; \\ 1/(\mathrm{n}+2) & \text{if } x = \frac{1}{n} \text{ for } n \geq 2; \\ x & \text{otherwise.} \end{cases}$$

This function is bijective. (Thanks to Zander Hill for this example.)

Next consider the function $x \mapsto e^x$, which produces a bijective correspondence between $\mathbb{R}$ and $(0, \infty)$.

Finally consider the function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, which is also bijective. This demonstrates how all of the open intervals are equivalent. $\square$

Proposition 15. $|\mathbb{R}| = |\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)|$.

Proof. Again let $O = (0,1)$. Since $|\mathbb{R}| = |O|$, it suffices to prove that the cardinality of O equals that of $\mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$.

First construct a function

$$f : \mathcal{F}(\mathbb{N}, \mathbb{Z}_2) \to O \quad \text{by} \quad f(a_i) = \sup \Big\{ \sum_{i=1}^{k} \frac{a_i}{10^i} \mid k \in \mathbb{N} \Big\}.$$

This function is injective.

Next consider that $\delta_2 : O \to \mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$ is injective.

By the Schoeder-Bernstein theorem, there exists a bijective function $O \to \mathcal{F}(\mathbb{N}, \mathbb{Z}_2)$. $\square$

8. Cardinal Numbers

Let U be a set; we refer to U as a *universal set*, and assume that U contains $\mathbb{R}$.

Let A and B be sets. We say that A and B have the same *cardinality* if there exists a bijective function between them. If A and B have the same cardinality, we write $A \sim B$. Then $\sim$ is a relation on $\mathcal{P}(U)$.

Proposition 16. *The relation $\sim$ is an equivalence relation on $\mathcal{P}(U)$.*

We shall call the equivalence classes of the relation the *cardinal numbers in U*. Let $\beth$ denote the set of cardinal numbers in U. If $A \subset U$, the equivalence class to which it belongs is denoted $|A|$, and is called the *cardinality* of A.

Define a relation $\leq$ on $\beth$ by

$$|A| \leq |B| \Leftrightarrow \exists \text{ injective } f : A \to B;$$

where $A, B \subset U$ are representatives of the cardinal numbers $|A|$ and $|B|$ respectively.

Proposition 17. *The relation $\leq$ on $\beth$ is well defined.*

That is, let $A_1, A_2, B_1, B_2 \subset U$ such that $A_1 \sim A_2$ and $B_1 \sim B_2$, and such that $|A_1| \leq |B_1|$. Show that $|A_2| \leq |B_2|$.

9. Schroeder-Bernstein Theorem

Lemma 1 (Banach's Lemma)**.** *Let X and Y be sets. and let $f : X \to Y$ and $g : Y \to X$ be injective functions. There exist subsets $A \subset X$ and $B \subset Y$ such that $f(A) = B$ and $g(Y \setminus B) = X \setminus A$.*

Proof. Fix the following objects:

- Let X and Y be sets.
- Let $f : X \to Y$ and $g : Y \to X$ be injective functions.
- Let $h = g \circ f$.
- Let $C_0 = X \setminus g(Y)$.
- Let $C_n = h(C_{n-1})$, for each $n \in \mathbb{N}$.
- Let $A = \cup_{n=0}^{\infty} C_n$.
- Let $B = f(A)$.

It suffices to show that $g(Y \setminus B) = X \setminus A$.

Claim 1: $h(A) \subset A$.
Let $a_0 \in h(A)$. Then $a_0 = h(a_1)$ for some $a_1 \in A$. By definition of A, $a_1 \in C_n$ for some $n \in \mathbb{N}$. Then $a_0 \in C_{n+1}$. Thus $a_0 \in A$.

Claim 2: $g(Y \setminus B) \subset X \setminus A$.
We want to select an arbitrary $y_0 \in Y \setminus B$ and show that g sends it into $X \setminus A$. Let $x_0 \in g(Y \setminus B)$. Then there exists $y_0 \in Y \setminus B$ such that $g(y_0) = x_0$. Suppose bwoc that $x_0 \in A$. Since $x_0 \in g(Y)$, $x_0 \notin C_0$, so $x_0 \in C_n$ for some $n > 0$. Since $C_n = h(C_{n-1})$, there exists $x_1 \in C_{n-1}$ such that $h(x_1) = x_0$. So $g(f(x_1)) = x_0$. Since g is injective, $f(x_1) = y_0$. But $x_1 \in A$, so $y_0 \in B$. This is a contradiction. Thus $x_0 \notin A$, so $x_0 \in X \setminus A$. Since x_0 was chosen arbitrarily, $g(Y \setminus B) \subset X \setminus A$.

Claim 3: $g(Y \setminus B) \supset X \setminus A$.
We want to select an arbitrary $x_0 \in X \setminus A$ and find $y_0 \in Y \setminus B$ which g sends to it. Let $x_0 \in X \setminus A$. Since $C_0 \subset A$, then $x_0 \in X \setminus C_0$. That is, $x_0 \in g(Y)$, so there exists $y_0 \in Y$ such that $g(y_0) = x_0$. Suppose bwoc that $y_0 \in B$. Then there exists $x_1 \in A$ such that $f(x_1) = y_0$. Thus $h(x_1) = x_0$, so $x_0 \in h(A)$. Since $h(A) \subset A$, $x_0 \in A$, which is a contradiction. Thus $y_0 \notin B$, so $x_0 \in g(Y \setminus B)$. Since x_0 was chosen arbitrarily, $X \setminus A \subset g(Y \setminus B)$. □

Theorem 3 (The Schroeder-Bernstein Theorem)**.** *Let X and Y be sets. If there exist injective functions $f : X \to Y$ and $g : Y \to X$, then there exists a bijective function $h : X \to Y$.*

Proof. Let A and B be sets as specified by the lemma. Let $V = X \setminus A$ and $W = Y \setminus B$. Then $f \restriction_A : A \to B$ is bijective, and $g \restriction_W : W \to V$ is bijective. Let $r = (g \restriction_W)^{-1}$. Then $r : V \to W$ is bijective. Thus define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ r(x) & \text{if } x \in V. \end{cases}$$

□

Corollary 1. *Let X and Y be sets. If there exist surjective functions $f : X \to Y$ and $g : Y \to X$, then there exists an bijective function $h : X \to Y$.*

Proof. This follows immediately by combining the Schroeder-Bernstein Theorem with the Axiom of Choice. $\square$

Corollary 2. *Let X and Y be sets. The following conditions are equivalent:*

- $|X| = |Y|$*;*
- $\exists$ *a bijective function* $X \to Y$*;*
- $\exists$ *injective functions* $X \to Y$ *and* $Y \to X$*;*
- $\exists$ *surjective functions* $X \to Y$ *and* $Y \to X$*.*

Proposition 18. *Show that* $(\beth, \leq)$ *is an ordered set.*

Proof. To show this, one must show that $\leq$ is a total order relation on $\mathcal{P}(U)$. The proof of symmetry uses the Schroeder Bernstein Theorem, and the proof of definiteness requires the Axiom of Choice. $\square$

The total order relation $\leq$ on $\beth$ naturally leads to the following definitions for derived relations on $\beth$:

- $|A| \geq |B| \Leftrightarrow |B| \leq |A|$;
- $|A| < |B| \Leftrightarrow \neg(|A| \geq |B|)$;
- $|A| > |B| \Leftrightarrow \neg(|A| \leq |B|)$.

10. Cardinal Arithmetic

Let A and B be sets. We define the sum, product, and exponentiation of cardinal numbers to match that of finite numbers.

Define
$$|A| + |B| = |(A \times \{0\}) \cup (B \times \{1\})|.$$
Note that even if $A \cap B$ is nonempty, $A \times \{0\}$ and $B \times \{1\}$ are disjoint sets. So if A is any set with m elements and B is any set with n elements, then $(A \times \{0\}) \cup (B \times \{1\})$ is a set with $m + n$ elements.

Define
$$|A| \cdot |B| = |A \times B|.$$
Again, if A and B are finite with m and n elements respectively, then $A \times B$ has mn elements.

Define
$$|A|^{|B|} = |\mathcal{F}(B, A)|,$$
where $\mathcal{F}(B, A)$ denotes the set of all functions from B to A. This again agrees with the finite case.

We have seen that for any set X, there does not exist a surjective function from X to its power set $\mathcal{P}(X)$. Thus $|X| < |\mathcal{P}(X)|$. We have also seen that $|\mathcal{P}(X)| = |\mathcal{F}(X, \mathbb{Z}_2)|$, which can be written as $|\mathcal{P}(X)| = 2^{|X|}$.

Department of Mathematics, BASIS Scottsdale
Email address: paul.bailey@basised.com